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Note

Lattices generated by skeletons of reflexive polytopes

Christian Haase, Benjamin Nill

Research Group Lattice Polytopes, FU Berlin, Arnimallee 3, 14195 Berlin, Germany

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Abstract

Lattices generated by lattice points in skeletons of reflexive polytopes are essential in determining the fundamental group and integral cohomology of Calabi–Yau hypersurfaces. Here we prove that the lattice generated by all lattice points in a reflexive polytope is already generated by lattice points in codimension two faces. This answers a question of John Morgan.

Keywords: Lattice polytopes; Calabi–Yau manifolds; Mirror symmetry

1. Introduction and main result

Since its introduction by Victor Batyrev in [1], dual pairs of reflexive polytopes have been used to successfully construct mirror pairs of smooth Calabi–Yau varieties as resolutions of non-degenerate anticanonical hypersurfaces in Gorenstein toric Fano varieties, see, e.g., [3].

Recall that a reflexive polytope is an n-dimensional lattice polytope $P \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ for a lattice $M \cong \mathbb{Z}^n$ such that P contains the origin in its interior and the polar polytope $P^* = \{x: \langle x, y\rangle \geqslant -1 \text{ for all } y \in P\}$ is also a lattice polytope with respect to the dual lattice $N = M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. There is an associated pair $\mathbb{P}_{\Sigma_P, M}$, $\mathbb{P}_{\Sigma_{P^*}, N}$ of Gorenstein toric Fano varieties, where Σ_P is just the fan of cones over the faces of P, see, e.g., [1,7].

For a given reflexive polytope $P \subseteq M_{\mathbb{R}}$ there exist finitely many choices of lattices such that P is reflexive with respect to this lattice. Obviously the minimal one of these is the lattice Λ_0 generated by the vertices of P. More generally we define

E-mail addresses: christian.haase@math.fu-berlin.de (C. Haase), nill@math.fu-berlin.de (B. Nill).

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$\Lambda_k :=$ lattice generated by all M-points in the k-skeleton of P,

where the k-skeleton is the union of k-dimensional faces.

Due to [7, Lemma 1.17] and [5, 3.2] we see that for $k = 0, 1, 2$ the quotient group M/Λ_k equals the fundamental group π_1 of the union of all $\leqslant k+1$-codimensional torus orbits in $\mathbb{P}_{\Sigma_P,M}$.

Returning to the relevance of these lattices in mirror symmetry, Victor Batyrev and Maximilian Kreuzer [2] and independently Charles Doran and John Morgan [4] studied the algebraic topology of a projective crepant resolution X of an $(n-1)$-dimensional non-degenerate anticanonical Calabi–Yau hypersurface in $\mathbb{P}_{\Sigma_P,M}$. For $n = 3, 4$ the quotient group M/Λ_{n-2} is precisely the fundamental group $\pi_1(X)$ of X [2, Corollary 1.9]. Moreover for $n = 4$ both groups showed that the torsion group of $H^2(X;\mathbb{Z})$ equals $\mathrm{Hom}(M/\Lambda_{n-2}, \mathbb{Q}/\mathbb{Z})$ and the torsion group of $H^3(X;\mathbb{Z})$ is isomorphic to $\mathrm{Hom}(\wedge^2 M/M \wedge \Lambda_{n-3}, \mathbb{Q}/\mathbb{Z})$, cf. [2, Corollary 3.9] and [4, Corollary 2.22]. Mirror symmetry should exchange these torsion parts of the integral cohomology of X and a mirror X^*. Hence, this yields the surprising isomorphism $M/\Lambda_2 \cong \wedge^2 N/N \wedge \Lambda_1^*$, where Λ_1^* is the sublattice of N generated by lattice points in edges of P^*. This conjecture was confirmed using the classification [6] of Maximilian Kreuzer and Harald Skarke.

These results and open questions motivate further investigation of the lattices Λ_k for an n-dimensional reflexive polytope P. Here we first note the following two observations (see, e.g., [7, Section 1]):

- Since there are no non-zero lattice points in the interior of P, we have $\Lambda_{n-1} = \Lambda_n$.
- If there exists a crepant toric resolution of $\mathbb{P}_{\Sigma_P,M}$, then the boundary ∂P contains a lattice basis, hence $\Lambda_{n-1} = M$. This holds for $n = 2, 3$.

For $n = 2$ there are precisely three isomorphism classes of reflexive polytopes with $\Lambda_{n-2} = \Lambda_0 \neq M$, here Λ_0 has index 2, 2, and 3, see Fig. 1.

As noted by Batyrev/Kreuzer and Doran/Morgan, for $n = 3$ a non-degenerate Calabi–Yau hypersurface X in a crepant resolution of $\mathbb{P}_{\Sigma_P,M}$ is a smooth $K3$-surface. So the fundamental group $\pi_1(X) \cong M/\Lambda_{n-2}$ is trivial, and $\Lambda_{n-2} = \Lambda_{n-1}$. We do not know of a combinatorial proof of this result in the literature.

At PCMI 2004, John Morgan asked the first author for a combinatorial proof that $\Lambda_{n-2} = \Lambda_{n-1}$ would also hold for $n = 4$. This was confirmed by the classification of all four-dimensional reflexive polytopes [6]. Doran/Morgan and Batyrev/Kreuzer have listed all 16 isomorphism classes of four-dimensional reflexive polytopes with $\Lambda_{n-2} \neq M$. However in contrast to $n = 3$ not even an algebro-geometric proof seemed to be known.

The goal of this article is to provide a purely convex-geometric proof valid in arbitrary dimension $n > 2$. (Figure 1 shows all "counter-examples" for $n = 2$.)

Theorem 1. *If $n \geqslant 3$, then $\Lambda_{n-2} = \Lambda_{n-1}$.*

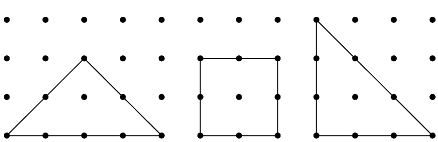

Fig. 1. Reflexive polygons with $\Lambda_{n-2} \neq M$.

The main idea of the proof is to show that any lattice point inside a facet can be obtained from lattice points on lower-dimensional faces. For this we exploit a partial addition property on the set of lattice points in P, observed by the second author in [7, Proposition 4.1]. This method was applied in [8] to the set $\mathcal{R}$ of lattice points inside the facets of P in order to investigate the automorphism group of the Gorenstein toric Fano variety $\mathbb{P}_{\Sigma_{P^*},N}$. The elements of $\mathcal{R}$ are precisely the associated Demazure roots.

2. Proof of the theorem

Let $P \subseteq M_{\mathbb{R}}$ be an n-dimensional reflexive polytope with boundary ∂P and vertices $\mathcal{V}(P)$. We denote by $\langle \cdot, \cdot \rangle$ the non-degenerate symmetric pairing of the dual lattices M, N.

For the proof we will use the following notions.

Definition 2.

- We denote by $\mathcal{R}$ the set of *Demazure roots* of P, i.e., the set of lattice points in the interior of facets of P.
- For $x \in \mathcal{R}$ we denote by $\mathcal{F}_x$ the unique facet of P containing x, and by η_x the unique inner normal of $\mathcal{F}_x$, i.e., $\eta_x = \eta_{\mathcal{F}_x} \in N$ s.t. $\langle \eta_x, \mathcal{F}_x \rangle = -1$.
- A pair of roots $x, y \in \mathcal{R}$ is called *orthogonal*, if $\langle \eta_x, y \rangle = 0 = \langle \eta_y, x \rangle$. In this case $x + y \in M \cap \mathcal{F}_x \cap \mathcal{F}_y$ (see the next result).

The following lemma is the engine running our proof. It has a purely combinatorial proof itself.

Lemma 3. *(See Nill [8, Lemma 4.8 and Corollary 4.9].) Let $x, y \in \partial P \cap M$ with $y \neq -x$ such that x, y are not contained in a common facet. Then there is a unique $z(x, y) = ax + by \in \partial P$ with $a, b \in \mathbb{Z}_{>0}$ such that $z(x, y)$ belongs to a common facet with x and with y.*

We have $a = 1$ or $b = 1$. Furthermore, if $x \in \mathcal{R}$, then $a = \langle \eta_x, y \rangle + 1$.

Figure 2 illustrates the situation of Lemma 3. The custom tailored version for our purposes talks about the point $p(x, y) := z(x, y) - x$. It reads as follows.[1] We abbreviate $\Lambda := \Lambda_{n-2}$.

Corollary 4. *Let $x \in \mathcal{R} \setminus \Lambda$, $y \in \partial P \cap \Lambda$ with $y \notin \mathcal{F}_x$. Then $\langle \eta_x, y \rangle \geqslant 1$, $p(x, y) := \langle \eta_x, y \rangle x + y \in \mathcal{R} \setminus \Lambda$ is a root orthogonal to x, and $y \in \mathcal{F}_{p(x,y)}$.*

Proof. Due to $-x \notin \Lambda$, we have $y \neq -x$, so let $z := z(x, y) = ax + by$ as in Lemma 3. We have $z \in \partial F_x \cap M$, hence $z \in \Lambda$. If $a = 1$, i.e., $\langle \eta_x, y \rangle = 0$, then $x = z - by \in \Lambda$, a contradiction. Therefore, we have $a \geqslant 2$, i.e., $\langle \eta_x, y \rangle \geqslant 1$. Hence $b = 1$, i.e., $z = ax + y$. Since $x = z - p(x, y) \notin \Lambda$, we get $p(x, y) \notin \Lambda$, so $p(x, y) \in \mathcal{R}$. This implies $y, z \in \mathcal{F}_{p(x,y)}$, and the roots $x, p(x, y)$ are orthogonal. $\square$

Proof of Theorem 1. Let $n \geqslant 3$, and let $x_1 \in \mathcal{R}$ arbitrary. We have to show $x_1 \in \Lambda$. Assume not, that is, $x_1 \notin \Lambda$.

[1] Of course, this corollary talks about the empty set as we aim to show.

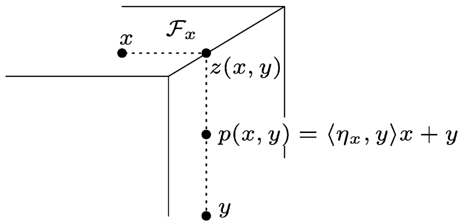

Fig. 2. $z(x, y)$ and $p(x, y)$.

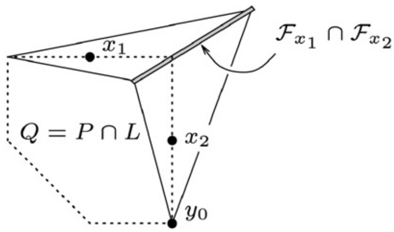

Fig. 3. $\mathcal{V}(P) \subset (\mathcal{F}_{x_1} \cap \mathcal{F}_{x_2}) \cup L$.

Let $y_0 \in \mathcal{V}(P) \setminus \mathcal{F}_{x_1}$ be a vertex of P outside $\mathcal{F}_{x_1}$, in particular, $y_0 \in \Lambda$. Corollary 4 provides $x_2 := p(x_1, y_0) \in \mathcal{R} \setminus \Lambda$. The points x_1 and x_2 span a 2-dimensional linear space L.

There are two cases: either all vertices of P belong to $(\mathcal{F}_{x_1} \cap \mathcal{F}_{x_2}) \cup L$ or not.

- $\mathcal{V}(P) \subset (\mathcal{F}_{x_1} \cap \mathcal{F}_{x_2}) \cup L$: See Fig. 3. The intersection $Q := P \cap L$ is a reflexive lattice polygon, since any vertex of Q that is not on $\mathcal{F}_{x_1} \cap \mathcal{F}_{x_2}$ is a vertex of P, and the other vertex $x_1 + x_2$ is integral, too. The part of ∂Q outside $\mathcal{F}_{x_1} \cup \mathcal{F}_{x_2}$ belongs to the 1-skeleton of P. As P contains the origin as the only lattice point in its interior, the endpoints of a primitive segment form a lattice basis for $L \cap M \ni x_1$. So $x_1 \in \Lambda_1 \subseteq \Lambda$, a contradiction. (This is where we use $n \geqslant 3$!)
- $\mathcal{V}(P) \not\subset (\mathcal{F}_{x_1} \cap \mathcal{F}_{x_2}) \cup L$: There are two subcases:
  - $\mathcal{V}(P) \subset \mathcal{F}_{x_1} \cup \mathcal{F}_{x_2} \cup L$: This is the most complicated case, see Fig. 4. Using Corollary 4 we can construct new roots from old ones. Because there are only finitely many roots, a maximal criminal argument yields the desired contradiction.

    As anchors for the construction we need a pair of vertices $y_1 \in \mathcal{F}_{x_1} \setminus \mathcal{F}_{x_2}$ and $y_2 \in \mathcal{F}_{x_2} \setminus \mathcal{F}_{x_1}$, such that one of the two sets $\{x_1, y_1, y_2\}$ or $\{x_2, y_1, y_2\}$ is linearly independent. (Assume no such pair exists. Then every vertex $y_1 \in \mathcal{F}_{x_1} \setminus \mathcal{F}_{x_2}$ has to lie in L, because otherwise $\{x_2, y_1, y_0\}$ would be linearly independent. Hence, every vertex $y_2 \in \mathcal{F}_{x_2} \setminus \mathcal{F}_{x_1}$ is in L, too, because otherwise $\{x_1, y_1, y_2\}$ would be linearly independent. Hence, $\mathcal{V}(P) \subset (\mathcal{F}_{x_1} \cap \mathcal{F}_{x_2}) \cup L$, a contradiction.)

    We may assume that $\{x_1, y_1, y_2\}$ is linearly independent. In order to identify the maximal criminal, choose $u \in N_{\mathbb{R}}$ so that $\langle u, x_1\rangle = \langle u, y_2\rangle = 0$ and $\langle u, y_1\rangle = 1$. Set $A := \{r \in \mathcal{F}_{x_1} \cap \mathcal{R} \setminus \Lambda\colon \langle \eta_{x_2}, r\rangle = 0\}$. Since $x_1 \in A$, we have $A \neq \emptyset$, so there exists $r \in A$ with $\langle u, r\rangle$ maximal (in particular $\geqslant \langle u, x_1\rangle = 0$).

    We will now construct an $r' \in A$ with higher u-value. Corollary 4 yields $k_1 := \langle \eta_{x_2}, y_1\rangle \geqslant 1$ and $k_2 := \langle \eta_{x_1}, y_2\rangle \geqslant 1$. Furthermore $q := p(r, y_2) = k_2 r + y_2 \in \mathcal{R} \setminus \Lambda$ and $\langle \eta_{x_1}, q\rangle = 0$ because of $\eta_{x_1} = \eta_r$. Since $\langle \eta_{x_2}, r\rangle = 0$ and $y_2 \in \mathcal{F}_{x_2}$, we get $q \in \mathcal{F}_{x_2}$. Corol-

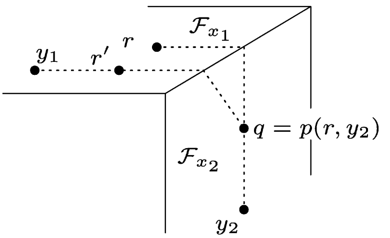

Fig. 4. Ping-pong.

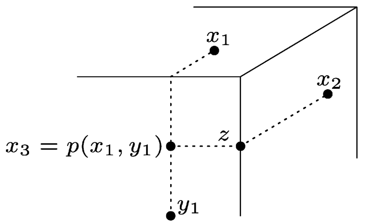

Fig. 5. Finish.

lary 4 implies again $r' := p(q, y_1) = k_1 q + y_1 \in \mathcal{R} \setminus \Lambda$, $\langle \eta_{x_2}, r' \rangle = 0$. Since $\langle \eta_{x_1}, q \rangle = 0$ and $y_1 \in \mathcal{F}_{x_1}$, we get $r' \in \mathcal{F}_{x_1}$. Hence $r' \in A$. Since $r' = k_1(k_2 r + y_2) + y_1$, we have $\langle u, r' \rangle = k_1 k_2 \langle u, r \rangle + 1 > \langle u, r \rangle$, a contradiction.

- $\mathcal{V}(P) \not\subset \mathcal{F}_{x_1} \cup \mathcal{F}_{x_2} \cup L$: See Fig. 5. Let $y_1 \in \mathcal{V}(P)$ outside $\mathcal{F}_{x_1} \cup \mathcal{F}_{x_2} \cup L$. Corollary 4 yields $x_3 := p(x_1, y_1) \in \mathcal{R} \setminus \Lambda$, and x_3 does not belong to $\mathcal{F}_{x_2}$ or L. By Lemma 3, $z := z(x_2, x_3) = a x_2 + b x_3 \in \partial P \cap \Lambda$ with $\langle \eta_{x_1}, z \rangle = b \langle \eta_{x_1}, x_3 \rangle = 0$, a contradiction to Corollary 4. □

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